

HEAVINESS IN CIRCLE ROTATIONS

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ABSTRACT. We are concerned with describing the structure of the set of points in the unit interval which, when subjected to rotation by irrational α modulo one, for all finite portions of the orbit contain at least as many points in the bottom half of the interval as in the top half. Specifically, an inductive procedure for describing the set based on the continued fraction expansion of α is developed, leading into a discussion of the Hausdorff dimension of this set. Depending on the parameter α , all possible dimensions may be achieved, and the essential infimum (with respect to α) of this dimension is positive.

1. INTRODUCTION AND BACKGROUND

We will call a point $x \in [0, 1)$ *heavy for* $[0, 1/2]$ *relative to rotation by* α if the sequence $n\alpha + x$ modulo one for $n = 0, 1, 2, \dots$ contains in every initial finite portion as many values in $[0, 1/2]$ as it does in its complement. Our interest is in describing the set of heavy points, given some specified α .

1.1. An Overview of Heaviness. All values considered herein should be considered modulo one. Heaviness may be viewed as recurrence of the set A back to itself (under rotation by α in the 1-torus) in terms of Schnirelmann density: x is heavy if

$$SD(\{\chi_{[0,1/2]}(x + n\alpha)\}) = \inf_{n \in \mathbb{N}} \frac{1}{n} \left(\sum_{k=0}^{n-1} \chi_{[0,1/2]}(x + k\alpha) \right) \geq \frac{1}{2}.$$

Heaviness is generally expressed in the context of dynamical systems: set $X = \mathbb{R}/\mathbb{Z} = [0, 1)$, $T(x) = R_\alpha(x) = x + \alpha \pmod{1}$, and $f(x) = \chi_{[0, \frac{1}{2}]}(x) - \chi_{(\frac{1}{2}, 1)}(x)$. Then by defining

$$S_n(x) = \sum_{i=0}^{n-1} f(T^i x)$$

we have $|S_{n+1}(x) - S_n(x)| = 1$, and the sums increase when $n\alpha + x \in A$, and decrease when $n\alpha + x \notin A$. The *heavy set for* A *relative to* T ,

$$H_T^A(\mathbb{N}) = \{x \in I : S_n(x) \geq 0, \quad \forall n \in \mathbb{N}\},$$

is nonempty [8] [6]. For an overview of heavy sets in general, see [8]. Heaviness in the case $\alpha \in \mathbb{Q}$ involves the study of periodic systems, appropriately viewed as heaviness in finite systems. In this context, heaviness may also be viewed as a generalization of *Lyndon words*, an object of study in combinatorics and computer science (see [5]). We abbreviate $H_{R_\alpha}^A(\mathbb{N}) = H_\alpha$. At a few points, we will be concerned with

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heaviness relative to a different set A' , in which case we consider the function $f(x) = \chi_{A'}(x) - \mu(A')$, and we write $H_\alpha^{A'}$.

The structure of the heavy set depends on the parameter α . Specifically, we will see that the structure of H_α is given by a constructive procedure based on the continued fraction expansion of α . We define the function $\varphi : [0, 1] \rightarrow [0, 1]$ by:

$$\varphi(\alpha) = \dim_H(H_\alpha),$$

where \dim_H is Hausdorff dimension. Then we have the following:

Theorem. Given any open $U \subset I$, $\varphi : U \rightarrow [0, 1]$ is surjective.

Theorem. For Lebesgue almost-every $\alpha \in [0, 1]$, $\varphi(\alpha) \geq c$, where $c > 0$ is an explicit constant which is independent of α .

In general, we can define heaviness for a sequence of points $\{\omega_i\}_{i=0}^\infty$, we let

$$S_n = \sum_{i=0}^{n-1} f(\omega_i),$$

and define the sequence to be heavy for f if $(S_n - n \int f d\mu) \geq 0$ for all $n \in \mathbb{N}$. By definition, the sequence $\{T^i \omega\}_{i=0}^\infty$ is heavy for f if and only if $\omega \in H_T^f(\mathbb{N})$. See [1] for one investigation of the sequence $\{i\alpha\}$ relative to the interval $[0, \frac{1}{k}]$; this situation will be revisited in Appendix A. The following lemma will be indispensable, and the simple proof may be found in [9]:

Lemma 1.1.1. *Let $\{\Omega, \mu, T\}$ be a probability measure preserving system (meaning $\{\Omega, \mu\}$ is a probability space, and T a transformation on Ω which preserves μ), and let $\omega \in \Omega$ and $f \in L^1(\Omega, \mu)$ be of mean zero. Assume that for some $i < j \in \mathbb{N}$:*

$$\min_{0 \leq k < j-i} \sum_{n=i}^{i+k} (f(T^n \omega)) = \sum_{n=i}^{j-1} f(T^n \omega) = 0.$$

Then $\omega \in H_T^f(\mathbb{N})$ if and only if the sequence $\omega, T(\omega), \dots, T^{i-1}\omega, T^j\omega, T^{j+1}\omega, \dots$ is heavy for f .

Remark 1.1.2. In our situation, a ‘hit’ to A followed by a ‘miss’ can be ignored. Letting A represent a hit to A and B a miss, the sequence

$$AABAABBA = A(AB)(A(AB)B)A,$$

for example, may be reduced to AA by successive removal of the pair AB . Consider the clear relation to the Catalan numbers (when considered as the number of nested sets of parentheses) for our situation; heaviness in general may be considered a dynamical-systems interpretation of the *ballot counting problem* [10].

Recall the following standard definition:

Definition 1.1.3. Let $\{\Omega, \mu, T\}$ be a probability measure preserving system, and let $A \subset \Omega$ be measurable. Then by the Poincaré Recurrence Theorem, for almost all $\omega \in A$, we may define

$$n(\omega) = \min\{n \in \mathbb{N} : T^n(\omega) \in A\},$$

and the *induced map* on A is given by

$$T_A(\omega) = T^{n(\omega)}(\omega).$$

1.2. Review of Continued Fractions and Hausdorff Dimension. The function $\varphi(\alpha)$ will be seen to be highly dependent on the continued fraction expansion of α , so we present here a brief overview of the theory of continued fractions and fractal dimension. We use *standard continued fraction notation*, where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for $i \in \mathbb{N}$:

$$[a_0; a_1, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}} = \frac{p_n}{q_n},$$

$$[a_0; a_1, a_2, \dots] = \lim_{n \rightarrow \infty} [a_0; a_1, a_2, \dots, a_n].$$

For a comprehensive coverage of the theory of continued fractions, see [4]. Every irrational α has a unique infinite continued fraction representation, and $q_i = a_i q_{i-1} + q_{i-2}$, $p_i = a_i p_{i-1} + p_{i-2}$. We define the *integer* and *fractional* parts of $x \in \mathbb{R}$, respectively, as

$$[x] = \max\{n \in \mathbb{Z} : n \leq x\}, \quad \{x\} = x - [x].$$

As we are only concerned with $\alpha \in (0, 1)$, the term a_0 is always zero, so we abbreviate

$$\alpha = [0; a_1, a_2, \dots] = [a_1, a_2, \dots].$$

Lemma 1.2.1. *Let $\alpha = [a_1, a_2, \dots]$. Then:*

$$\begin{aligned} \frac{a_2}{a_1 a_2 + 1} &< \alpha < \frac{a_2 + 1}{a_1(a_2 + 1) + 1}, \\ \frac{1}{a_1(a_2 + 1) + 1} &< 1 - a_1 \alpha < \frac{1}{a_1 a_2 + 1}, \\ \text{if } a_1 = 1 : (1 - \alpha) &= [a_2 + 1, a_3, a_4, \dots], \\ \text{if } 1 \neq a_1 = 2n + 1 : \frac{1}{2n + 1} &< 1 - 2n\alpha < \frac{2}{2n + 2}, \\ \left\{ \frac{1}{\alpha} \right\} &= [a_2, a_3, \dots], \\ p_2 &= \left\lfloor \frac{\alpha}{1 - q_1 \alpha} \right\rfloor, \\ \{q_2 \alpha\} &= q_2 \alpha - p_2. \end{aligned}$$

Proof. Only the last two merit discussion; the first five are elementary properties of continued fractions. Note that $a_1 = q_1$, $a_2 = p_2$, and use the first two inequalities to see:

$$\frac{\frac{a_2}{a_1 a_2 + 1}}{\frac{1}{a_1 a_2 + 1}} < \frac{\alpha}{1 - q_1 \alpha} < \frac{\frac{a_2 + 1}{a_1(a_2 + 1) + 1}}{\frac{1}{a_1(a_2 + 1) + 1}}$$

and reduce the fractions to yield the first equality.

For the last equality, we also use that $q_2 = a_1 a_2 + 1$:

$$\begin{aligned} \frac{a_2}{\alpha} &< q_2 < \frac{1}{1 - q_1 \alpha} \\ a_2 &< q_2 \alpha < \frac{\alpha}{1 - q_1 \alpha} \\ p_2 &< q_2 \alpha < p_2 + 1. \end{aligned}$$

□

For every $m \in \mathbb{N}$, the following limit exists and is a finite constant for Lebesgue almost-all α [4, Theorem 35]:

$$k_m = \lim_{n \rightarrow \infty} \left(\prod_{i=1}^n (a_i + m) \right)^{\frac{1}{n}},$$

where k_0 is *Khinchin's constant*. We state the following:

$$0.9877 < \log(k_0) < 0.9878, \quad 1.4097 < \log(k_1) < 1.4098.$$

We omit the tedious work in establishing these bounds; the persistent reader may use elementary techniques in summation and standard software packages to obtain the desired error control.

We notate the Hausdorff and lower box dimensions of a set A by $\dim_H(A)$ and $\dim_b(A)$, respectively. See [2] for a more complete treatment of this material.

Let E_0 be a closed, connected interval, and define

$$E_0 \supset E_1 \supset E_2 \supset \dots$$

by requiring that each E_k be a disjoint union of closed intervals, such that each interval of E_k contains at least two intervals of E_{k+1} , and the maximal length of an interval in E_k tends to zero as k tends to infinity. Set $F = \bigcap_{k=1}^{\infty} E_k$.

Theorem 1.2.2 ([2], Example 4.6). *Suppose that each interval of E_{k-1} contains at least m_k intervals of E_k , and that the intervals of E_k are all separated by at least ϵ_k , where $0 < \epsilon_{k+1} < \epsilon_k$. Then*

$$(1) \quad \dim_H(F) \geq \liminf_{k \rightarrow \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \epsilon_k)}.$$

We may combine (1) with the fact that $\dim_H(A) \leq \dim_b(A)$ in general to derive the following:

Theorem 1.2.3. *If each interval of E_k contains exactly m_k intervals of E_{k+1} , each of which is of length δ_k times the length of the intervals in E_k , and furthermore the intervals of E_{k+1} are separated by gaps at least as large as the intervals of E_{k+1} , and the m_k grow ‘sub-factorially’ (meaning ‘slower than factorially,’ not ‘like the subfactorial numbers’):*

$$(2) \quad \forall \epsilon > 0, \quad \exists N : \left(k > N \Rightarrow m_{k+1} < \prod_{i=1}^k m_i^\epsilon \right),$$

then

$$\dim_H(F) = \dim_b(F) = \liminf_{k \rightarrow \infty} \frac{\log(m_1 \cdots m_k)}{-\log(\delta_1 \cdots \delta_k)}.$$

Proof. First, note that the growth condition on m_k implies that

$$\lim_{k \rightarrow \infty} \frac{\log m_{k+1}}{\log(m_1 \cdots m_k)} = 0,$$

which in turn implies that

$$\liminf_{k \rightarrow \infty} \frac{\log(m_1 \cdots m_k)}{-\log m_{k+1} - \log(\delta_1 \cdots \delta_k)} = \liminf_{k \rightarrow \infty} \frac{\log(m_1 \cdots m_k)}{-\log(\delta_1 \cdots \delta_k)}.$$

Each E_{k+1} may be exactly covered by $m_1 \cdots m_k$ intervals of length $\delta_1 \cdots \delta_k$, and so:

$$\liminf_{k \rightarrow \infty} \frac{\log(m_1 \cdots m_{k-1})}{-\log(m_k \delta_1 \cdots \delta_k)} \leq \dim_H(F) \leq \liminf_{k \rightarrow \infty} \frac{\log(m_1 \cdots m_k)}{-\log(\delta_1 \cdots \delta_k)} = \dim_b(F). \quad \square$$

Remark 1.2.4. For those $\alpha = [a_1, a_2, \dots]$ for which the geometric mean of the a_i exist, the a_i must grow subfactorially, and for almost all α , the geometric mean of the a_i is k_0 .

1.3. Two Specific Examples: $\alpha_1 = [0; 2, 2, 2, \dots]$ and $\alpha_2 = [0; 3, 2, 2, 2, \dots]$. A demonstration of the explicit difference between the two rotations $\alpha_1 = [2, 2, 2, \dots]$ and $\alpha_2 = [3, 2, 2, 2, \dots]$ will make the general process clearer in §2. While $\sqrt{2} - 1 = \alpha_1$ and $(2 - \sqrt{2})/2 = \alpha_2$ have very similar continued fraction expansions, we will see that the corresponding heavy sets have radically different structure.

Example 1.3.1. $H_{\sqrt{2}}$ is a perfect set (closed with no isolated points) of Hausdorff dimension

$$\frac{\ln 3}{-\ln(3 - 2\sqrt{2})} = .623 \dots$$

We provide a rough plot of the heavy set $H_{\sqrt{2}}$ (see Figure 1).

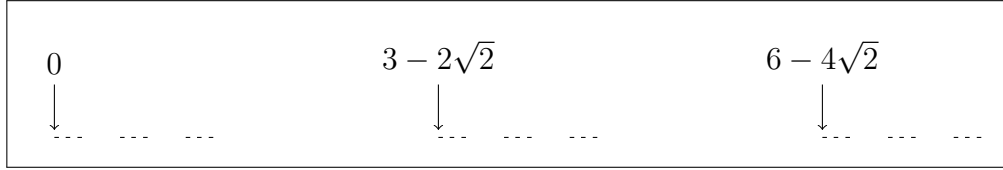


Figure 1: The set H_α with a few labelled points, where $\alpha = [2, 2, 2, \dots] = \sqrt{2} \bmod 1$. Note the fractal structure: three copies scaled by $3 - 2\sqrt{2}$.

As we are only concerned with rotation modulo one, set $\alpha = [2, 2, 2, \dots] = \sqrt{2} - 1$ and $T(x) = x + \alpha \bmod 1$. First, consider the interval $I' = [0, 1 - 2\alpha)$. One may verify that the induced transformation (recall Definition 1.1.3) $T_{I'}$, after rescaling by a factor of $1/(1 - 2\alpha)$, is again rotation by α (see Figure 2).

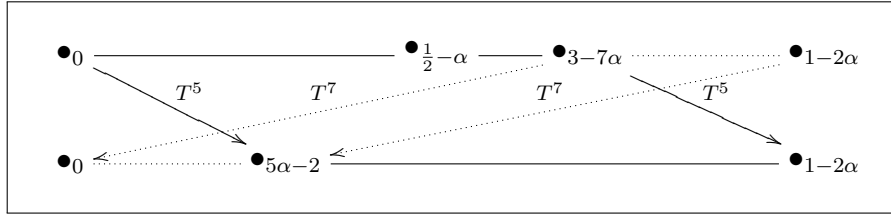


Figure 2: The induced map is rotation by $\frac{5\alpha - 2}{1 - 2\alpha} = \alpha$. Note that $5 = q_2$ and $7 = q_2 + q_1$.

On this subinterval I' , one may also verify that the points in the interval $[0, \frac{1}{2} - \alpha]$ return with the orbit segment AAB (by which we mean $x \in A$, $Tx \in A$, and $T^2x \notin A$), while the points in the intervals $(\frac{1}{2} - \alpha, 5 - 7\alpha)$ and $[5 - 7\alpha, 1 - 2\alpha)$ return with

ABB and $ABBAB$, respectively. Let $[0, \frac{1}{2} - \alpha] = A'$. By applying Lemma 1.1.1 we see that $x \in [0, 1 - 2\alpha)$ is in H_T^A if and only if it is also in $H_{T'}^{A'}$, because by freely removing the pair AB , the orbits reduce to A on A' and B on $I' \setminus A'$. So we have in fact shown that within the subinterval A' , *being heavy for A under T is equivalent to being heavy for A' under T'* , because the characteristic function of A under first return map reduces via Lemma 1.1.1 to the characteristic function of A' . Also, $T'_I = R_\alpha$, after rescaling, so we in fact see that $H_T^A \cap [0, \delta] = \delta H_T^A$, where $\delta = 1 - 2\alpha$.

Furthermore, the interval $[\delta, 2\delta)$ takes 2 steps to orbit into $[0, \delta)$ with segment AB , and the interval $[2\delta, 1/2)$ orbits into $[0, \delta)$ in 4 steps with segment $ABAB$. Again by Lemma 1.1.1, we have that $H_T^A \cap [\delta, 2\delta) = \delta H_T^A + \delta$, and similarly for the interval $[2\delta, 3\delta)$, where we note that the portion $(1/2, 3\delta)$ will not be heavy (as it is not in A), but the corresponding portion inside $[0, \delta)$ is also not heavy (it is not in $[0, \delta/2]$, the scaled down copy of A). Also, the entire interval $(\frac{1}{2} - \alpha, 1)$ fails to be heavy.

Overall, therefore, the heavy set $H_{\sqrt{2}}$ consists of three evenly spaced copies of itself scaled by a factor of $\delta = 1 - 2\alpha$. Now we apply Theorem 1.2.3, with $m_i \equiv 3$ and $\delta_i \equiv 1 - 2\alpha$.

Example 1.3.2. Fix $A = [0, \frac{1}{2}]$, $\alpha = [3, 2, 2, 2, \dots] = (2 - \sqrt{2})/2$, and $T(x) = x + \alpha$ (modulo one). Then H_α is countably infinite, with only one accumulation point, given by $\alpha/2$. The set H_α is therefore of Hausdorff dimension zero.

We will use many of the same tricks as in the previous example, but with very different results. See Figure 3 for a picture of H_α lying inside the interval $[0, \frac{1}{2}]$. We will again consider the induced map on the interval $[0, 1 - 2\alpha)$. However, as $1 - 2\alpha > \alpha$, the return times are much smaller: the interval $[0, 1 - 3\alpha)$ returns in one step with orbit A , the interval $[1 - 3\alpha, \frac{1}{2} - \alpha]$ returns in three steps with orbit AAB , and $(\frac{1}{2} - \alpha, 1 - 2\alpha)$ returns in three steps with orbit ABB (see Figure 4).

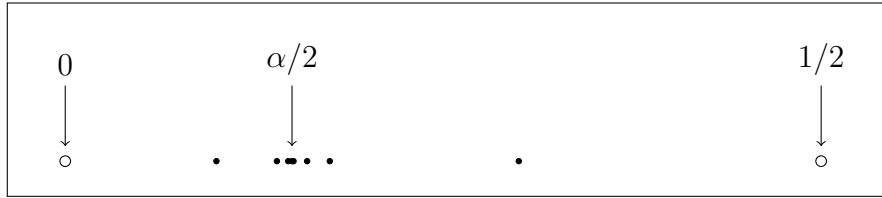


Figure 3: An approximate drawing of the set H_α sitting inside the interval $[0, \frac{1}{2}]$, where $\alpha = [3, 2, 2, \dots] = (2 - \sqrt{2})/2$. Note the sparse structure: isolated points converging towards a single accumulation point.

Lemma 1.1.1 applies again, then, to say that

$$H_\alpha \cap [0, 1 - 2\alpha) = H_{T'}^{[0, \frac{1}{2} - \alpha]}.$$

However, in the previous example, we were able to use the fact that $T_{I'} = R_\alpha$ to show that the heavy set in the induced map was a rescaled copy of the original heavy set. In this case, however, $T_{I'} = R_{1-\alpha}$, so we instead get

$$H_{T'}^{[0, \frac{1}{2} - \alpha]} = -\delta H_T^A + \left(\frac{1}{2} - \alpha\right),$$

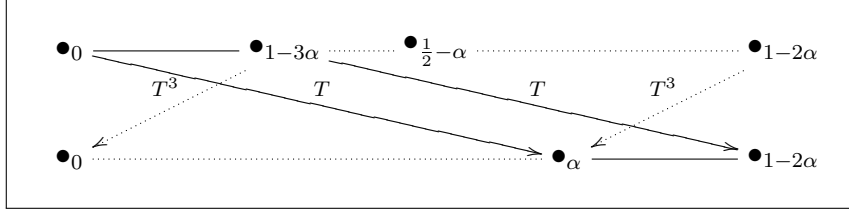


Figure 4: The induced map scales to rotation by $\frac{\alpha}{1-2\alpha} = 1 - \alpha$, slightly different than in Example 1.3.1, Figure 2. Note the return times are $1 = q_0$ and $3 = q_1$.

where $\delta = 1 - 2\alpha$ again. This fact results from considering rotation by $1 - \alpha$ to be equivalent to reversing orientation on our interval I' , and rotating by α . In this manner, whatever happens to a point x under R_α is duplicated in the behavior of $\frac{1}{2} - x$ under rotation by $1 - \alpha$.

Another crucial difference with the previous example concerns the fact that in Example 1.3.1, we were able to find a total of three copies of the heavy set. In this case, however, the smaller interval $[\delta, \frac{1}{2}]$ orbits into $[0, \frac{1}{2} - \delta]$ with orbit segment AB , so $H_\alpha \cap [\delta, \frac{1}{2}] = H_\alpha \cap [0, \frac{1}{2} - \delta] + \delta$. However, the entire interval $(\frac{1}{2}, 1)$ is clearly not heavy, so we only have a single “complete” heavy island, and then a “broken,” smaller heavy island. See Figure 5.

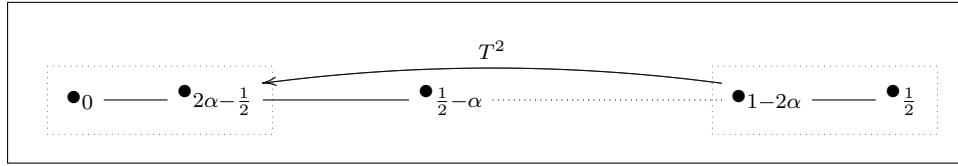


Figure 5: The smaller, “broken island” $[\delta, 1/2]$ will exhibit the same heaviness properties as the interval $[0, 1/2 - \delta]$, a subinterval of the “complete island” ($\delta = 1 - 2\alpha$). As the induced map on this complete island reverses orientation, most of the interval $[0, 1/2 - \delta]$ fails to be heavy, as it is now mirroring the behavior of the ‘top’ of the new system.

The behavior of that broken heavy island is dependent on the behavior of the interval $[0, \frac{1}{2} - \delta]$, because it orbits into it with initial segment AB , which we may freely ignore. However, recall that $T_{I'} = R_{1-\alpha}$, so the induced map will create the same picture, *but with reversed orientation*. Therefore, the induced map will create a broken heavy island on the interval $[0, (\frac{1}{2} - \delta)^2]$. Furthermore, the behavior in $[0, \frac{1}{2} - \delta]$ reduces to only considering this smaller interval, as $\frac{1}{2} - \delta < \delta - \delta^2$, so the interval $[0, \frac{1}{2} - \delta]$ does not contain any of the unbroken heavy island formed in the induced map.

Each time we pass to an induced map, we create a broken heavy island scaled down by a factor of $\frac{1}{2} - \delta < 1$; any given broken heavy island will decay to an isolated point. Furthermore, since we only create one unbroken island at each stage, scaled down by $\delta < 1$, there is only a single heavy point which results from intersecting the unbroken

islands. The exact value of $\alpha/2$ as the value of this accumulation point is not difficult to compute as the limit point of the endpoints of the unbroken islands.

2. THE FOUR ESSENTIAL CASES

In the following four cases, we investigate how the structure of H_α is affected by the initial terms of the continued fraction expansion of α . Recall that $I = [0, 1)$, $A = [0, \frac{1}{2}]$. Regardless of whether $a_1 = 2n$ or $a_1 = 2n + 1$, we also write $I' = [0, 1 - 2n\alpha]$, $A' = [0, \frac{1}{2} - n\alpha]$ (so $n = [\frac{a_1}{2}]$), and we will be building the induced map $T_{I'}$. Our process will always consider which subintervals of A maintain nonnegative sums for larger times n . In §3 and §4, we will employ this process to compute $\varphi(\alpha)$.

2.1. The simplest case to consider is that of $a_1 = 1$. In this event, $\alpha > \frac{1}{2}$, and $1 - \alpha = [a_2 + 1, a_3, a_4, \dots]$. As rotation by α is symmetric with rotation by $1 - \alpha$, we see that a point ω is heavy for A relative to rotation by α if and only if $\frac{1}{2} - \omega$ is heavy for A relative to rotation by $1 - \alpha$. So, as in Example 1.3.2, by reversing our orientation, we may consider rotation by $[a_2 + 1, a_3, a_4, \dots]$ without generating any subintervals.

2.2. In the event that $a_1 = 2n$ and $a_3 \geq 2$, the following order of elements is verifiable by direct computation:

$$\begin{aligned} 0 &< \frac{1}{2} - n\alpha < 1 - 2n\alpha < \frac{3}{2} - 3n\alpha < 2 - 4n\alpha < \dots \\ \dots &< a_2 - a_1 a_2 \alpha < \frac{2a_2 + 1}{2} - n(2a_2 + 1)\alpha < \frac{1}{2} - (n-1)\alpha < (a_2 + 1) - a_1(a_2 + 1)\alpha < 1. \end{aligned}$$

Note that A' is exactly the closed bottom half of I' , just as A is the closed bottom half of I . Now, let $I'_k = [k - 2nk\alpha, (k+1) - 2n(k+1)\alpha]$, so that $I' = I'_0$, and let $A'_k = [k - 2nk\alpha, \frac{2k+1}{2} - n(2k+1)\alpha]$, so that $A' = A'_0$.

Lemma 2.2.1. *If $x > \frac{1}{2} - (n-1)\alpha$, then $x \notin H_\alpha$. Furthermore, if $1 - 2n\alpha < x < \frac{1}{2} - (n-1)\alpha$, then x is heavy for A if and only if $\{x + 2n\alpha\} = (x + 2n\alpha - 1)$ is heavy for A .*

Proof. First, assume that $\frac{1}{2} > x$. The orbit of x begins with no more than $n-1$ hits to A (as $x > \frac{1}{2} - (n-1)\alpha$), followed by at least n hits to $I \setminus A$ (as $x < \frac{1}{2}$), so that x is not heavy for A . If $x > \frac{1}{2}$, then x fails to be heavy for A trivially. Now, if $1 - 2n\alpha < x < \frac{1}{2} - (n-1)\alpha$, we see that the first $2n$ points in the orbit of x are n consecutive hits to A , followed by n consecutive hits to $(I \setminus A)$. By Lemma 1.1.1, we may ignore this string of the orbit when determining heaviness, so we see that x is heavy for A if and only if $\{x + 2n\alpha\}$ is heavy for A . Note also that as $\frac{1}{2} > x > 1 - 2n\alpha$ was assumed, we know that $\{x + 2n\alpha\} \in A$. \square

Lemma 2.2.2. *If $x \in A'$, then x remains heavy for A until it returns to I' , at which point it returns with a net weight of $+1$. Conversely, those $x \in (I' \setminus A')$ fail to be heavy for A , and they return to I' with a net weight of -1 , and they at no point through their return have a net weight of less than -1 .*

Proof. Let $x \in A'$. Then as $0 < x < \frac{1}{2} - n\alpha$, the orbit of x begins with an A , and then $1 - 2n\alpha < x + \alpha < \frac{1}{2} - (n-1)\alpha$. By Lemma 2.2.1, we may now apply R_α a

total of $2n$ times to the point $y = x + \alpha$, producing the orbit segment $A^n B^n$; the initial string of n hits to A is due to $y < \frac{1}{2} - (n-1)\alpha$, itself due to $x < \frac{1}{2} - n\alpha$. Now, if $\{y + 2n\alpha\} > 1 - 2n\alpha$, we apply R_α another $2n$ times, again using Lemma 2.2.1. We continue to apply R_α in blocks of $2n$ applications until we land back in I' , at which point we may not apply the same lemma again. However, we have shown the desired conclusion; the orbit of x contains an initial A , and then $y = x + \alpha$ has an orbit consisting of repetitions of the block $A^n B^n$ (which we may ignore by Lemma 1.1.1) until it returns to I' .

For those $x \in (I' \setminus A')$, the only difference is that $x + \alpha = y > \frac{1}{2} - (n-1)\alpha$. So the orbit begins with $A^n B^{n+1}$, but after that produces the same blocks $A^n B^n$. So by applying Lemma 1.1.1, this orbit segment may be reduced to the single extra ‘miss,’ so that those $x \in (I' \setminus A')$ fail to be heavy, but also return to I' with a net weight of -1 , and this net loss of one is the most extra misses accumulated at any time. \square

Overall, we may decompose the set A (those points which remain heavy for A through at least one step) into a_2+1 equally spaced subintervals of length $(1-2n\alpha)|A|$, the intervals A'_k for $k = 0, 1, \dots, a_2$; the string of inequalities at the beginning of this case includes the fact that A'_{a_2} is the last A'_k before the cutoff value of $\frac{1}{2} - (n-1)\alpha$, which does not interrupt this subinterval. The heaviness behavior in each ‘subisland’ A'_k reduces to the behavior in the interval A' . As we know that A' remains heavy for A until it returns to I' , we now compute the induced transformation on the interval I' :

Lemma 2.2.3. *The induced transformation on I' is rotation by $\alpha' = [a_3, a_4, \dots]$.*

Proof. It follows from the elementary theory of continued fractions that the minimal m such that $m\alpha < 1 - 2n\alpha = 1 - q_1\alpha$ is $m = q_2 = 2na_2 + 1$. From this observation, one may quickly deduce that the induced transformation on I' is given by Figure 6:

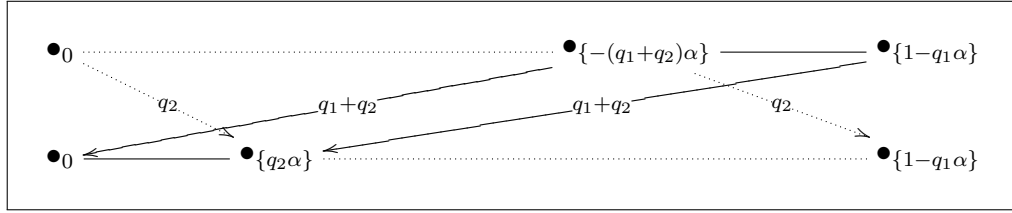


Figure 6: The first case $a_1 = 2n$ for building the first return map.

After rescaling, then, the induced transformation is rotation by the amount $\{q_2\alpha\}/(1 - q_1\alpha)$. Using Lemma 1.2.1:

$$\begin{aligned}
 \alpha' &= \frac{\{q_2\alpha\}}{1 - q_1\alpha} = \frac{q_2\alpha - p_2}{1 - q_1\alpha} \\
 &= \frac{(p_2q_1 + 1)\alpha - p_2}{1 - q_1\alpha} = \frac{\alpha - (1 - q_1\alpha)p_2}{1 - q_1\alpha} \\
 &= \frac{\alpha}{1 - q_1\alpha} - \frac{p_2 - q_1p_2\alpha}{1 - q_1\alpha} = \frac{\alpha}{1 - q_1\alpha} - \left[\frac{\alpha}{1 - q_1\alpha} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \left\{ \frac{\alpha}{1 - q_1 \alpha} \right\} = \left\{ \frac{1}{\frac{1}{\alpha} - q_1} \right\} \\
&= \left\{ \frac{1}{\left\{ \frac{1}{\alpha} \right\}} \right\} = [a_3, a_4, \dots]. \quad \square
\end{aligned}$$

To summarize, then, the interval A may be decomposed into $a_2 + 1$ equally spaced subintervals, each of whose heaviness behavior is controlled by the interval I' and its induced transformation. This induced transformation is another rotation, and as the interval A' represents a net $+1$ and the interval $(I' \setminus A')$ represents a net -1 , heaviness in A is mirrored on each of these subintervals A'_k by heaviness for A' under the induced rotation by α' . This point deserves repetition, as it is the central observation for deducing the structure of our heavy set: heaviness for A each interval A'_k is now controlled by heaviness for A' under the induced rotation $R_{\alpha'}$ on the interval I' . As A' is again half of I' , we have returned to the original problem of describing heaviness for a closed half-interval, except that we now rotate by $\alpha' = [a_3, a_4, \dots]$, and the manner in which A' decomposes into heavy sub-intervals will be mirrored on each A'_k .

2.3. In the case when $a_3 = 1$, but a_1 is still even, the exact same argument applies, with one crucial adjustment. In this case, we have:

$$(a_2 + 1) - 2n(a_2 + 1)\alpha < \frac{1}{2} - (n - 1)\alpha < \frac{2a_2 + 1}{2} - (2na_2 + 1)\alpha.$$

So, the interval $A'_{a_2+1} = [a_2 + 1 - 2n(a_2 + 1)\alpha, \frac{2a_2+1}{2} - (2na_2 + 1)\alpha]$ would be another heavy subinterval, but it is interrupted by that critical cutoff value of $\frac{1}{2} - (n - 1)\alpha$. So, the same arguments will show that the intervals A'_k behave the same way as in the case $a_3 = 1$ for $k = 0, 1, \dots, a_2$, but in A'_{a_2+1} , only the ‘broken island’ $[a_2 + 1 - 2n(a_2 + 1)\alpha, \frac{1}{2} - (n - 1)\alpha]$ will remain heavy for A . Letting z be the length of this partial subinterval, we note that $0 < z < |A'|$, and that heaviness in this partial subinterval will mirror heaviness in the interval $[0, z] \subset A'$. The induced map on I' is again given by rotation by $[a_3, a_4, \dots]$.

2.4. The only case remaining is when $a_1 = 2n + 1$, $n \neq 0$. We state the following easily verifiable inequalities:

$$0 < \frac{1}{2} - n\alpha < 1 - 2n\alpha < \frac{1}{2} - (n - 1)\alpha < \frac{3}{2} - 3n\alpha.$$

The same techniques as in §2.2 and §2.3 show that heaviness for A reduces to considering heaviness for $A' = [0, \frac{1}{2} - n\alpha]$ in $I' = [0, 1 - 2n\alpha]$, under the induced transformation on I' . Now, however, $|I'| > \alpha$, so the return times are much smaller. See Figure 7. As $\alpha < 1 - 2n\alpha$, the induced transformation is given by rotation by $\alpha/(1 - 2n\alpha)$:

$$\begin{aligned}
\frac{\alpha}{1 - 2n\alpha} &= \frac{1}{\frac{1}{\alpha} - 2n} \\
&= \frac{1}{2n + 1 + [a_2, a_3, \dots] - 2n} \\
&= [1, a_2, a_3, \dots].
\end{aligned}$$

So, in this case, we have produced a single complete heavy island scaled in length by $1 - 2n\alpha$, as well as a single broken heavy island of intermediate length, and the new rotation to consider is given by $\alpha' = [1, a_2, a_3, \dots]$.

2.5. Summary. Refer to Table 1 to summarize the decomposition of the heavy set, where $\alpha = [a_1, a_2, \dots]$.

Case	Complete Islands:	Scaled by:	Broken Island?	New α :
$a_1 = 1$	1	1	no	$[a_2 + 1, a_3, \dots]$
$a_1 = 2n, a_3 > 1$	$a_2 + 1$	$1 - 2n\alpha$	no	$[a_3, a_4, \dots]$
$a_1 = 2n, a_3 = 1$	$a_2 + 1$	$1 - 2n\alpha$	yes	$[a_3, a_4, \dots]$
$a_1 = 2n + 1, n \neq 0$	1	$1 - 2n\alpha$	yes	$[1, a_2, a_3, \dots]$

Table 1: A summary of how the heavy set for A decomposes based on α .

3. EXPLICIT HAUSDORFF DIMENSION COMPUTATIONS

This process of producing a nested sequence of subintervals nicely follows the construction needed in §1.2.3, so we may begin explicit computation of $\varphi(\alpha)$.

3.1. Surjectivity of the map φ . Recall that if $a_1 = 2n, a_3 > 1$, we produce evenly spaced subintervals of constant size, each of which will behave exactly as the others, and then apply a rotation by $[a_3, a_4, \dots]$. So, when the odd-indexed partial quotients of α are all even, the process is particularly nice, as $[a_3, a_4, \dots]$ will obey the same condition: $a_3 = 2m, a_5 > 1$. Among such α , the easiest to consider are those whose continued fraction expansion is of period two.

Lemma 3.1.1.

$$\text{Let } \alpha_{2n,m} = [2n, m, 2n, m, \dots] = \frac{-nm + \sqrt{nm(nm - 2)}}{2n}.$$

$$\text{Then } \varphi(\alpha_{2n,m}) = \frac{\log(m + 1)}{\log(1 + nm - \sqrt{nm(nm + 2)})}.$$

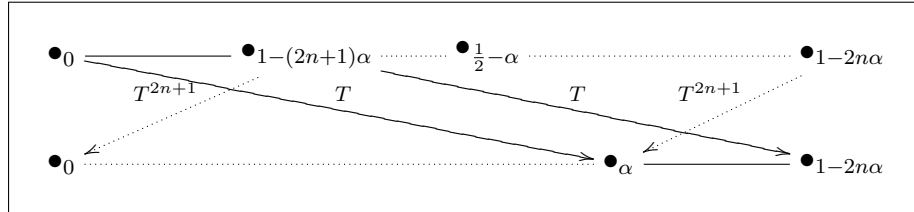


Figure 7: The induced map on I' in the event that $a_1 = 2n + 1$, where $n \neq 0$.

Proof. That $\alpha_{2n,m}$ is a quadratic irrational of the prescribed form is a simple computation. To determine the value $\varphi(\alpha_{2n,m})$, we note that in applying the techniques of §2, we will always have $a_1 = 2n$, $a_2 = m$, $a_3 = 2n \geq 2$. Therefore, our heavy set may be represented as an intersection of a nested sequence of intervals, where each interval breaks into $m + 1$ subintervals of length $1 - 2n\alpha_{2n,m} = 1 + nm - \sqrt{nm(nm + 2)}$, each of which is separated by gaps of the same length, and by Theorem 1.2.3,

$$\varphi(\alpha_{2n,m}) = \frac{\log(m + 1)}{\log(1 + nm - \sqrt{nm(nm + 2)})}. \quad \square$$

Corollary 3.1.2. *The values $\varphi(\alpha_{2n,m})$ as $n, m = 1, 2, \dots$ are dense in $[0, 1]$.*

Proof. Consider the function

$$f(x, y) = \frac{\log(y + 1)}{\log\left(1 + xy - \sqrt{xy(xy + 2)}\right)}$$

for $x, y \in \mathbb{R}^+$. Of course, $f(2n, m) = \varphi(\alpha_{2n,m})$ for $n, m \in \mathbb{N}$. The following may all be verified with elementary (if tedious) methods:

- For fixed x_0 , $\lim_{y \rightarrow \infty} f(x_0, y) = 1$.
- For fixed y_0 , $\lim_{x \rightarrow \infty} f(x, y_0) = 0$.
- $\partial f / \partial x < 0$, $\partial^2 f / \partial x^2 > 0$.

Now, fix $\epsilon > 0$, and pick N large enough so that $f(2, N) > 1 - \epsilon$. Then the sequence $f(2m, N)$ as $m = 1, 2, \dots$ monotonically decreases towards zero, with the gaps between successive elements shrinking, thereby forming an ϵ -dense subset of $[0, 1]$. As ϵ was arbitrary, the result is proved. \square

Theorem 3.1.3. *The function φ maps every open subset of I surjectively onto I .*

Proof. Let $\delta \in [0, 1]$, and by Corollary 3.1.2, pick $2n_i, m_i$ such that $\lim f(2n_i, m_i) = \delta$. Then let $\alpha = [2n_1, m_1, 2n_1, m_1, \dots, 2n_2, m_2, 2n_2, m_2, \dots]$ where each pair $2n_i, m_i$ is repeated $k(i)$ times. $k(i)$ is chosen to be large enough so that the entries of α satisfy the sub-factorial growth condition (2), so that the Hausdorff dimension of the resulting heavy set is equal to its lower box dimension. Furthermore let $k(i)$ be large enough so that as we remove successive pairs by applying our inductive process on α , the resulting rotations remain within 2^{-i} of α_{2n_i, m_i} for at least 2^i successive iterations, so that the \liminf in the box dimension computation is equal to the limit of our chosen $\varphi(\alpha_{2n,m})$. Carefully tracking the values $k(i)$ would be cumbersome, and would obscure the only relevant fact; they exist and are finite, so that for any $\delta \in [0, 1]$ we may construct α with $\varphi(\alpha) = \delta$.

To see that for any open U , φ surjectively maps U onto I , note that for any open $U \subset [0, 1]$, there is some string a_1, a_2, \dots, a_n such that all numbers whose continued fraction expansion begins with $[a_1, a_2, \dots, a_n]$ belong to U . While we cannot control how the heavy set decomposes as a result of these first n values, we may clearly control the rest of the entries to achieve any desired result. \square

Corollary 3.1.4. *If we define $\Phi(\alpha) = (x, y)$, where x and y are the lower and upper box dimensions of the heavy set, then Φ surjectively maps every open subset of the unit interval onto the entire possible image $\{(x, y) : 0 \leq x \leq y \leq 1\}$.*

Proof. By choosing two targets x and y , where $x \leq y$, we may alternate long stretches of the continued fraction expansion of α , again beginning arbitrarily deep in the expansion, so that the \limsup and \liminf of the dimension computation achieve our desired amounts. \square

4. ALMOST-SURE BOUNDS ON HAUSDORFF DIMENSION

Two inherent problems prevent us from directly computing $\varphi(\alpha)$, but we may address each issue to derive a meaningful lower bound. First, the lengths of the produced intervals will vary depending on the rotation amounts at each stage, which will require computing the exact form of a given continued fraction. We may easily find an lower bound on the length, however, relying on no more than two terms of the continued fraction expansion of α . Second, there is the possibility of ‘broken intervals’ being produced, of intermediate length, which may or may not disappear or decay to a single point (as in Example 1.3.2) in later steps. We will skirt this problem by ignoring such broken intervals completely, still maintaining a lower bound.

Letting n_i be a lower bound on the number of subintervals produced after i applications of the cases in §2, and letting l_i be a lower bound on the lengths of the intervals, we consider our starting α to be represented by a sequence of natural numbers - the continued fraction expansion. Set $n_1 = l_1 = 1$, and apply the following process, derived from the results of §2:

- (1) If $a_1 = 2n$ and $a_3 \geq 2$, let $n_{i+1} = (a_2 + 1)n_i$ and $l_{i+1} = ((a_1 + 1)(a_2 + 1))^{-1}$. Now consider the sequence a_3, a_4, \dots
- (2) If $a_1 = 2n$ and $a_3 = 1$, let $n_{i+1} = (a_2 + 1)n_i$, and $l_{i+1} = ((a_1 + 1)(a_2 + 1))^{-1}$. Now consider the sequence $a_4 + 1, a_5, \dots$
- (3) If $a_1 = 2n + 1$ and $n \neq 0$, let $n_{i+1} = n_i$ and $l_{i+1} = (a_1 + 1)^{-1}l_i$. Now consider the sequence $a_2 + 1, a_4, \dots$

We will refer to the preceding events as Case 1, 2, and 3, respectively. Note that we have ignored the possibility that our sequence ever begins with one. Without loss of generality, we consider only those $\alpha < \frac{1}{2}$, and we have already noted that we can safely ignore the possibility of changing orientation for these estimates; we automatically consider any sequence $1, a_2, a_3, \dots$ to be $a_2 + 1, a_3, a_4, \dots$

By Theorem 1.2.3, if our n_i grow sub-factorially (which is true for Lebesgue almost-every α , as previously remarked), then:

$$\liminf_{n \rightarrow \infty} \frac{\log n_i}{-\log l_i} \leq \varphi(\alpha).$$

Given a fixed irrational $\alpha = [a_1, a_2, \dots]$, let $\alpha_i = [a_{m(i)}, a_{m(i)+1}, \dots]$ be the number associated with applying cases 1-3 (as appropriate) i times to α . Note that $i + 1 \leq m(i) \leq 3i + 1$: if case 3 is applied, we remove only a single terms from the list, while if case 2 is applied, we remove three (case 1 removes two terms), and $m(0) = 1$.

Lemma 4.0.5. *Almost surely, we have both*

$$\liminf_{i \rightarrow \infty} \frac{\log n_i}{m(i)} > c_1$$

$$\limsup_{i \rightarrow \infty} \frac{-\log l_i}{m(i)} < c_2,$$

for finite positive constants c_1 and c_2 (the constants are independent of α).

Proof. First, note that $n_i \geq 2^{m'(i)}$, where $m'(i)$ is the number of times out of the first i applications of cases 1-3 that either case 1 or 2 was applied. So, we need only prove that almost-surely,

$$\liminf_{i \rightarrow \infty} \frac{m'(i)}{3i} > c.$$

We apply one of these cases when an even a_1 is followed by an even a_3 or when an odd a_1 is followed by an odd a_2 , where we are testing along an increasing subsequence of the natural numbers. Our inequality follows from the following fact [4, Ch. 12]:

$$\mathbb{P}(a_3 = 0 \pmod{2} | a_1 = 0 \pmod{2}) \geq \frac{1}{3} \mathbb{P}(a_3 = 0 \pmod{2}) = \frac{\log 2 - \log(\pi/2)}{3 \log 2},$$

$$\mathbb{P}(a_2 = 1 \pmod{2} | a_1 = 1 \pmod{2}) \geq \frac{1}{3} \mathbb{P}(a_3 = 1 \pmod{2}) = \frac{\log(\pi/2)}{3 \log 2}.$$

So, regardless of what has happened in the previous steps, the probability of encountering case 1 or 2 is at least $(\log 2 - \log(\pi/2))/(3 \log 2)$. Overall, then, $\liminf_{i \rightarrow \infty} \frac{\log n_i}{m_i} \geq (\log 2 - \log(\pi/2))/3$.

For the inequality involving l_i , we note that $-(1/m(i)) \log(l_i)$ is the logarithm of the geometric mean of a certain subsequence of the $(a_i + 1)$ with gaps between successive entries at most two. As the geometric mean of *all* $a_i + 1$ is almost-surely the constant k_1 , the limsup of the logarithm of the geometric mean along a syndetic subset of gap bounded by two is almost-surely no larger than $2 \log(k_1) \approx 2.818$. \square

We immediately arrive at our second major claim:

Theorem 4.0.6. *For Lebesgue almost-every $\alpha \in [0, 1]$:*

$$\varphi(\alpha) \geq \frac{\log 2 - \log(\pi/2)}{6 \log(k_1)} > .028.$$

4.1. Miscellaneous Results. Now consider the set

$$H_\alpha^*(\mathbb{N}) = \{x : S_n(x) > 0, \forall n \in \mathbb{N}\},$$

of those points for which a *strict* inequality is maintained. In each of the four cases ($a_1 = 2n$, etc) there was exactly *one* subinterval which maintained a strict inequality; recall that when more than one heavy island was produced (as happened in all cases except $a_1 = 1$), only the least island maintained a strict inequality, while all other complete or partial islands began their orbit with $A^n B^n$ (where $a_1 = 2n$ or $2n + 1$). Therefore, by intersecting the unique closed island which maintains a sum of at least $+1$, we see that for every $\alpha \notin \mathbb{Q}$, the set $H_\alpha^*(\mathbb{N})$ is a single point. As a corollary to this observation, we also note that for all $\alpha \notin \mathbb{Q}$, the set H_α is infinite. If x_0 is the unique ‘strictly heavy’ point, then there must be infinitely many N such that $S_m(x_0) = 1$; if not, the final such time would produce another strictly heavy point. Because the

sums never return to zero, however, each such time m corresponds to a (non-strictly) heavy point.

Similarly, the orbit of any heavy point must contain infinitely many other heavy points. However, it is worth noting that while the set of strictly heavy points is very small, a single point, the set of heavy points is generally fairly large, of positive Hausdorff dimension, so that ‘most’ heavy points do not belong to the orbit of this unique strictly heavy point. In Appendix A, we investigate one special case of identifying this unique strictly heavy point, first investigated in [1].

We also note that for almost every α , H_α contains arbitrarily long arithmetic progressions. For any fixed N , one must merely notice that proceeding through the four cases of §2 will yield $\alpha' = [2n, N, \dots]$ for some n at least once (in fact, infinitely many times) for almost every α , and at this stage, the heavy set will decompose into $N + 1$ equally spaced subislands, all of which will decompose identically in further steps, easily producing infinitely many arithmetic progressions of length $N + 1$ in the heavy set. We may also construct some α for which the heavy set is countably infinite, with any fixed $N = k_1 k_2 \dots k_j$ accumulation points organized in k_1 clusters, each with k_2 subclusters, each with k_3 subclusters, etc, where each $k_i \geq 2$: set

$$\alpha = [2, k_1 - 1, 2, k_2 - 1, \dots, 2, k_j - 1, 3, 2, 2, 2, \dots].$$

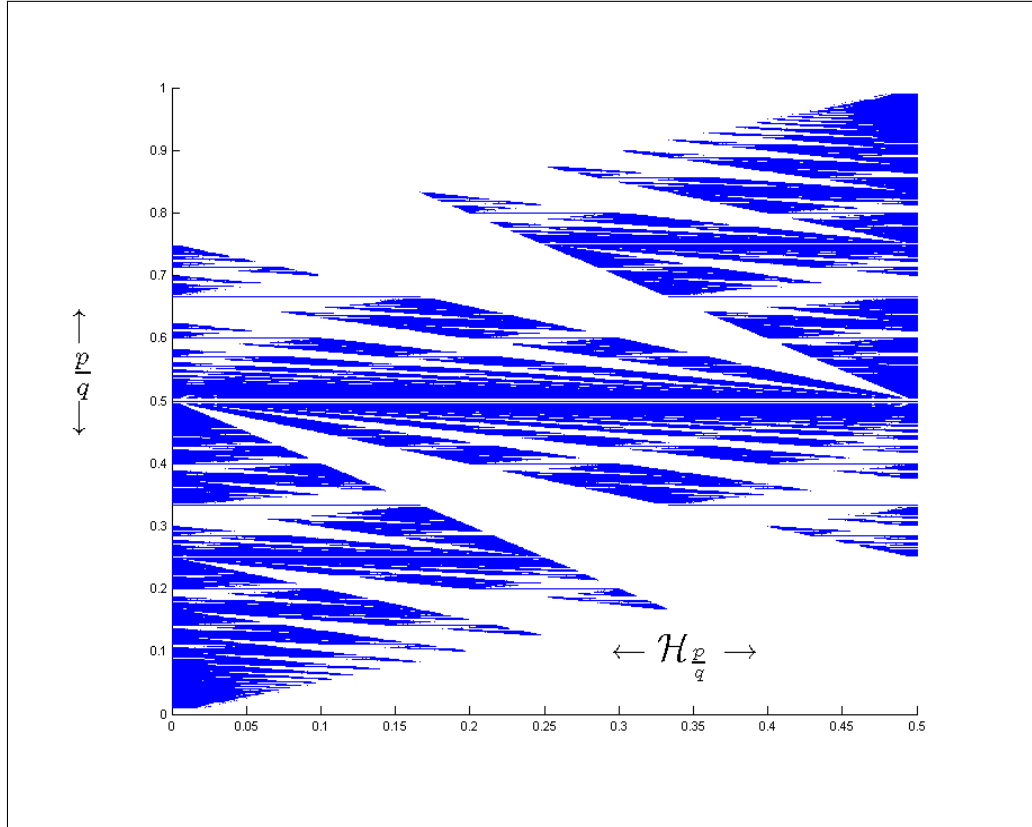


Figure 8: The heavy sets $H_{p/q}$ for all $q \leq 100$.

Then we create k_1 islands, each with k_2 subislands, etc, until finally reaching a rotation by $[3, 2, 2, \dots]$, which we have already seen in Example 1.3.2 will cause each remaining island to decay to exactly one accumulation point. However, the set H_α always contains exactly one solution to the equation $x - y = \alpha$. If both $x, y \in H_\alpha$, and $x = y + \alpha$, we see that y must be a strictly heavy point, of which there is only one. Similarly, the unique strictly heavy point x_0 always has $x_0 + \alpha \in H_\alpha$, so one solution always exists.

In conclusion, we present a picture (Figure 8). For $\alpha = p/q$, the set H_α is a finite collection of closed intervals. With the aid of a computer, we present a plot of $H_{p/q}$ along the x -axis for various p/q along the y -axis in Figure 8. Included are all p/q with $p < q$, $q \leq 100$. Besides the obvious relation between H_α and $H_{1-\alpha}$, the structure of this set is not clear. By Theorem 4.0.6, the Hausdorff dimension of this set is positive, but thus far there are only visual suggestions of self-similarity.

APPENDIX A. CONNECTIONS TO HEAVY SEQUENCES

We now mention a slight extension of the results of §2.2 and §2.4. For this section only, let $A = [0, \frac{1}{k}]$ for some fixed $k \in \mathbb{N}$, $I' = [0, 1 - nk\alpha]$, and $A' = [0, \frac{1}{k} - n\alpha]$. It is now necessary to let $f(\omega) = k\chi_A(\omega) - 1$. The proofs may be carried out exactly as in the original version (where Lemma 1.1.1 now allows us to remove the block AB^{k-1}), and we omit them:

Lemma A.0.1. *Let $a_1 = nk$. Then for $x \in A'$, heaviness under induced map $T_{I'}$ with respect to A' is equivalent to heaviness under T with respect to A , as in §2.2. $T_{I'}$ rescales to rotation by $[a_3, a_4, \dots]$. If $a_1 = nk + i$, where $n \neq 0$ and $1 \leq i \leq k-1$, then heaviness under T relative to A is again equivalent to heaviness under $T_{I'}$ relative to A' , and $T_{I'}$ rescales to rotation by $[i, a_2, a_3, \dots]$.*

We are now in a position to claim the following theorem:

Theorem A.0.2. *The sequence $\{i\alpha\}$ ($i = 1, 2, \dots$) is heavy for $[0, \frac{1}{k}]$ if and only if $a_{2i+1} = 0 \pmod k$ for all i .*

Proof. One direction is immediate from Lemma A.0.1. Assume that $\alpha = [a_1, a_2, \dots]$ where $a_{2i+1} = 0 \pmod k$. Then 0 remains heavy for A until it returns to I' and maintains one extra ‘hit,’ at which point heaviness is equivalent to heaviness under the induced transformation with respect to A' , but the induced transformation is rotation by $[a_3, a_4, \dots]$, which still has $a_{2i+1} = 0 \pmod k$. It follows that $0 \in H_\alpha$, or that the sequence $0, \alpha, 2\alpha, \dots \pmod 1$ is heavy for A . We have been tracking the orbit of zero, but it follows that as the orbit of zero always maintains one extra ‘hit,’ the orbit of α itself is heavy.

Now, let some $a_{2i+1} \neq 0 \pmod k$. We may assume without loss of generality that $a_1 < k$; the inductive process from Lemma A.0.1 will maintain heaviness until the first odd-indexed a_i which is not zero modulo k , at which point if it is larger than k , it will induce a rotation by an amount whose first partial quotient is less than k . So, let $\alpha = [i, a_2, a_3, \dots]$, where $i < k$. Then $\alpha > 1/k$, and the sequence is clearly not heavy. \square

Remark A.0.3. In [1], the same condition on α is shown to be equivalent to heaviness of the sequence $\{i\alpha\}$ ($i = 0, 1, 2, \dots$). The only difference is that for the sequence

beginning at $i = 0$, a strict inequality will be maintained. Also, in [1], rational α are considered as well as irrational, and the interval is open on the right (which will only affect the sequence for rational α), both of which are minor variations.

APPENDIX B. FRACTAL PROPERTIES OF DISCREPANCY SUMS

Let $\alpha \notin \mathbb{Q}$ as before, and let $f(x) = \chi_A(x) - \chi_{I \setminus A}(x)$, where $A = [0, 1/2]$ again, and continue to let

$$S_n(x) = \sum_{i=0}^{n-1} f(x + i\alpha).$$

Consider the function $\xi : \mathbb{N} \rightarrow \mathbb{Z}$ given by $\xi(n) = S_n(0)$. In his book [3], D. Hensley noted that when $\alpha = \sqrt{2}$, the sequence $\xi(n)$ has very regular structure. See Figure 9 for four images for several N , where $1 \leq n \leq N$.

Our techniques allow us to explain this structure, as well as describe an inductive process for producing these sums for all $\alpha \notin \mathbb{Q}$. Continuing with the specific case of $\alpha = \sqrt{2}$, note that we can just as well consider $\alpha = \sqrt{2} - 1 = [2, 2, 2, \dots]$. We

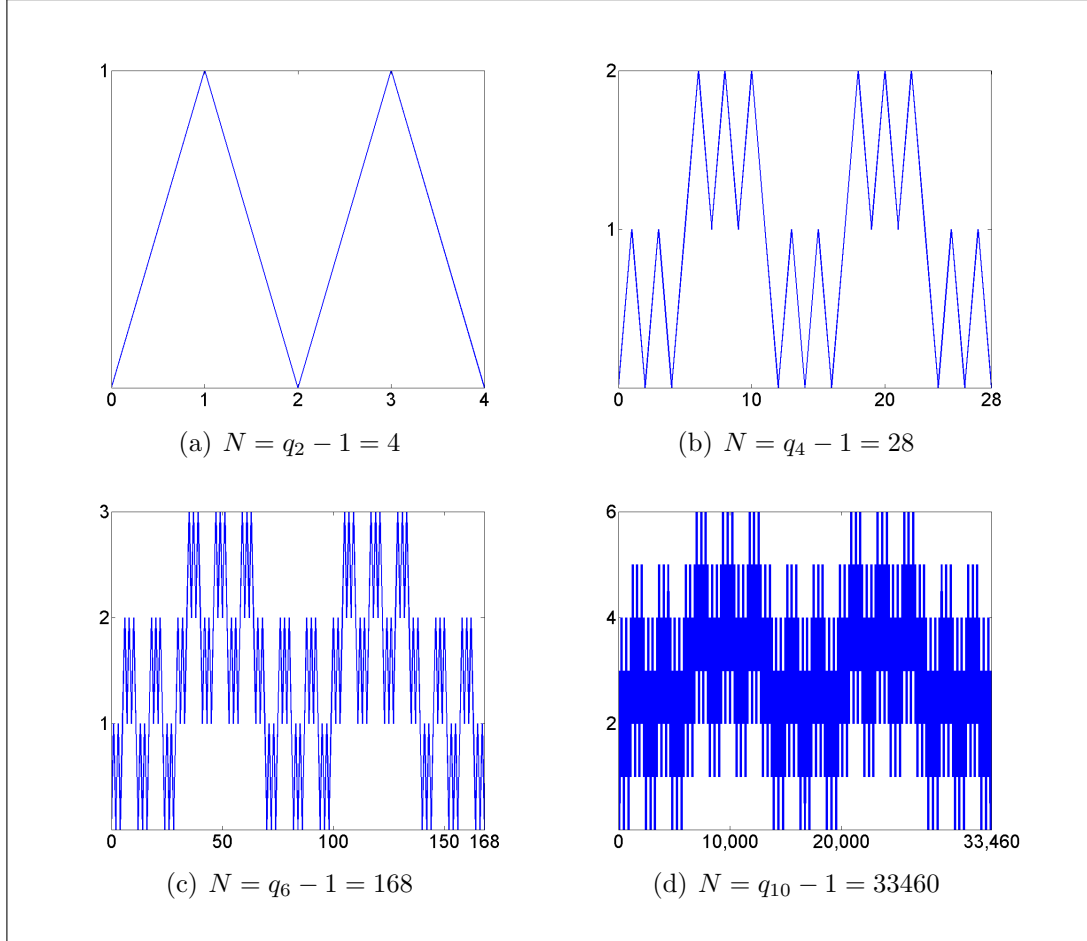


Figure 9: Linearly interpolated graphs of $\xi(n)$, $n = 0, 1, \dots, N$. Note the developing fractal-like structure and consistent nonnegativity.

have already noted in Example 1.3.1 that through the first q_2 steps, this sum remains nonnegative, and then returns to the interval A' , and that the induced map $T_{I'}$ will again be rotation by $\sqrt{2} - 1$. Each successive iteration of the induced map produces a string of either length q_2 or $q_2 + q_1$, with overall changes of ± 1 . These sums remain positive for all time (the sequence $\{n\sqrt{2}\}$ is heavy for A by Theorem A.0.2), but also contain a nicely iterated structure. Whatever blocks of up/down movement corresponds to the different possibilities represented by the induced map $T_{I'}$ (q_2 steps with a net $+1$, $q_2 + q_1$ steps with a net -1 , etc), these blocks will themselves be placed according to the map $T_{I'}$, which is our original rotation by $\sqrt{2}$.

This observation generalizes for arbitrary α , with the caveat that when α is not of the form $[2n_1, m_1, 2n_2, m_2, \dots]$, we will need to consider the situation of some step involving $\alpha' = [2n + 1, m, \dots]$ where $n \neq 0$. In this case, recall that our induced map $T_{I'}$ involved small return times (1 and q_1 , specifically), so that the process of pasting together blocks at this stage will produce proportionately large changes in the value $\xi(n)$; the overall change for each block is still ± 1 , but we have not enlarged the size of the blocks compared to the previous step. This generic situation explains the lack of such neatly arranged sequences $\xi(n)$ in general. Still, it is not difficult to see that for almost-all α , this process will result in considering $\alpha' = [2n, m, \dots]$ infinitely many times, so there are arbitrarily long segments which will have this highly ordered structure.

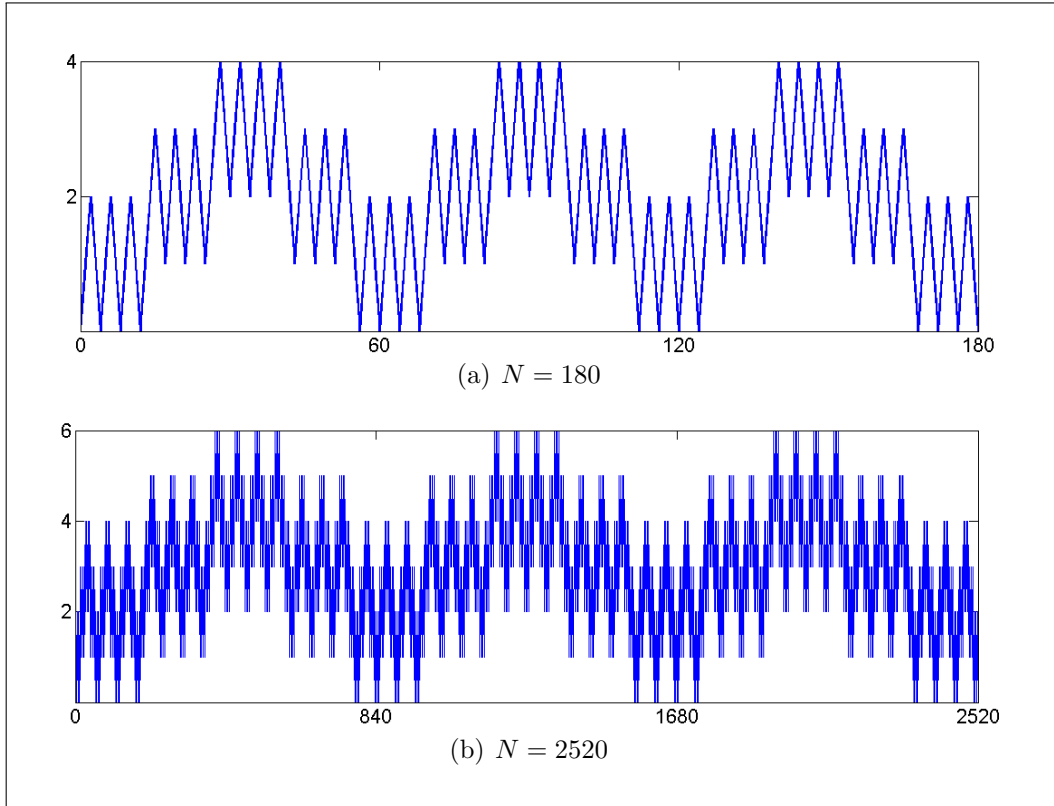


Figure 10: Linearly interpolated graphs of $\xi(n)$, $n = 0, 1, \dots, N$, where we have set $\alpha = [4, 3, 4, 3, \dots]$.

The developing structure first noticed for $\alpha = \sqrt{2}$ in [3] will certainly generalize to $\alpha = [2n, m, 2n, m, \dots]$. In fact, the process will apply to any quadratic α for which $\{n\alpha\}$ is heavy for A , but a small period and small values of n, m allow the structure to be clear within a reasonably small window of time. See Figure 10 for the discrepancy sums for $\alpha = \sqrt{3} - \frac{3}{2} = [4, 3, 4, 3, 4, 3, \dots]$. The role of $n = 2$ corresponds to a height in each small peak of $+2$, while $m = 3$ accounts for producing a sequence of three peaks before considering the induced structure.

ACKNOWLEDGEMENTS

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